SOME RESULTS OF THE PERTURBATION METHOD APPLIED TO TWO-DIMENSIONAL NON-DIVERGENT MODELS IN AN INCOMPRESSIBLE FLUID

BY

CHARLES ELLIS TILDEN

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PREFACE

This investigation was conducted at the United States Naval Postgraduate School, Monterey, California as the thesis requirement for the degree of Master of Science in Aerology.

For help and advice received in its preparation the author is indebted to Associate Professor G. J. Haltiner of the Postgraduate School.

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TABLE OF SYMBOLS AND ABBREVIATIONS

U	Undisturbed Zonal Wind Velocity
U_o	Minimum Undisturbed Zonal Wind Velocity
u	Perturbation Zonal Wind Velocity
v	Perturbation Meridional Wind Velocity
p	Perturbation Pressure
f	Coriolis Parameter
B	Variation of Coriolis Parameter with y
α	Wave Number
c	Wave Velocity
b	Parameter in Sinusoidal Velocity Profile
r	$+\left[\alpha^2 + \frac{B}{c-U}\right]^{1/2}$
K, a_n	Constants of Integration
k	Parameter in Probability Curve Velocity Profile
ρ	Density of Fluid
d	$c - U_o$
t	Time
i	$\sqrt{-1}$
ϕ	Latitude
ω	Angular Speed of the Earth
x	Axis Points East
y	Axis Points North
$'$	Denotes Differentiation with Respect to y

I. INTRODUCTION

In the attempt to subject the state of the atmosphere to a mathematical analysis, and in particular, to explain the growth and behavior of wave like disturbances in the westerlies, the meteorologist is confronted with two major problems:

(a) An imperfect knowledge of the instantaneous state of the atmosphere and

(b) The mathematical difficulties inherent in the attempt to integrate the equations of motion.

Adequate observational data is available to permit us to draw some conclusions as to the "average" conditions in the atmosphere, or, in the case of choosing a particular model to analyze, a "plausible" condition.

The mathematical problem is simplified by considering the effect of a small perturbation on a known zonal flow and neglecting all differentials in the resulting equations of higher order than the first. This results in a set of linear differential equations -- the perturbation equations. The mathematical details have, in most cases, been further reduced by making certain simplifying assumptions in the models. Although such assumptions introduce artificialities when compared with the atmosphere, they essentially resolve the problem to the study of dynamic instability -- that portion of the stability or instability contributed by the nature of the flow pattern itself. Kuo [4] states that, in studying two-dimensional nondivergent flow,

Similar results would be obtained for the very general case with vertical motion, solenoids, and divergence and convergence, if certain approximations are made in reducing the partial differential equations for a certain dependent variable to an ordinary differential equation.

The technique of using the perturbation equations is not new, having been used over fifty years ago by Helmholtz, Rayleigh and others. In 1928 Solberg showed that unstable waves can develop in a sloping surface of discontinuity between two parallel streams of different density, supporting the results of his synoptic studies of wave development on the polar front. In 1939 Rossby developed his widely known equation for the speed of long waves in the westerlies in the special case of constant zonal motion in a homogeneous incompressible atmosphere. Since that time, a great number of meteorologists have applied the theory to a variety of models. Machta [5], using Rossby's model, assumed a solution for the perturbation stream function which allowed for a tilt to the trough line. Haurwitz also used Rossby's model but allowed for the curvature of the earth and obtained only slight modification to Rossby's results. Charney [1] considered a special case of baroclinic three-dimensional flow.

Kuo [4] developed general stability criteria for a two-dimensional, barotropic nondivergent atmosphere using general, symmetric velocity profiles. Haurwitz [3] considered pure shearing waves between two constant zonal two-dimensional wind fields on a rotating planet. The case of a shear zone of finite width was considered neglecting the variation of the coriolis parameter with latitude. This problem in considered briefly in this paper.

It may be noted that in nearly all of the papers studied only a portion of the complete problem was investigated. Some are concerned with the question of whether or not a perturbation will be unstable. Others have eliminated

potentially unstable waves from their solutions (as will be done in this paper) and proceed to establish the speed with which the long stable waves will move. This approach is entirely reasonable because if the wave is unstable, the solution is valid only in the initial development and the significance of the real values of wave velocity is limited to this time. Also, as Charney [1] points out, the evaluation of the complex roots may present great difficulties. On the other hand, if the wave is essentially stable, the second order differentials which have been neglected in the perturbation equations will continue to be small and the value of wave velocity obtains is valid for an appreciable length of time.

In this paper, only the long, slow moving or retrograding waves are considered. The frequency equation resulting from the solution of the boundary systems determined by a sinusoidal velocity profile and a linear shear zone are determined. The frequency equation for a probability curve profile was not fully developed due to mathematical difficulties encountered in the application of one of the boundary conditions. This matter is discussed in some detail.

The roots of the frequency equation in the case of the sinusoidal velocity profile are evaluated (for a particular wave length) for a synoptic situation in which the horizontal wind velocity profile was in fair agreement with the model. The results are found to be of the right order of magnitude and show that there may be more than one possible value of wave speed under a given set of conditions.

We shall now consider three velocity profiles and derive the equations relating wave speed and wave number. The perturbation equations to be used, as developed by Haurwitz [2], assume incompressible flow. We shall further assume two-dimensional nondivergent flow. The variation of the coriolis parameter with latitude is assumed to be a linear function of y. (Appendix II). Under these conditions, the perturbation equations reduce to

$$v''(c-U) + v(U'' - \beta - \alpha^2 c + \alpha^2 U) = 0 \tag{1}$$

as derived in Appendix I.

A complete analysis of this equation must include a study of the behavior of v in the neighborhood of the singular point $c = U$. Such a point must occur if the wave is moving with a speed between U_0 and U_{max}. Kuo [4] has investigated this point for the case of general symmetric velocity profiles by including frictional terms in the above equation. These terms are small compared with the others except when $c - U$ is small.

He shows that if a singular point exists, a stable wave can exist only if $U'' - \beta = 0$ at some point on the velocity profile. If this condition is met, the speed of such a stable wave is the value of U at that point. Kuo calls this the critical velocity. Any wave moving with less than this velocity (but greater than U_0) will be unstable, while those moving with a velocity greater than the critical velocity will be damped. He further shows that for the long slow moving waves ($c < U_0$), the amplifying or dampening factors are small or zero and such waves are essentially neutral.

In the solutions to be discussed, no closed form was found and the power series solutions obtained are valid only for $(c - U) < 0$, the slow moving or retrograding waves.*

* Kuo [4] has shown that the minimum wave length of such waves is about 5200 km under typical shear conditions.

II. SINUSOIDAL VELOCITY PROFILE

The first problem to be considered is the solution of equation (1) where

$$U = U_0 + U_1 \sin by \qquad 0 \leq y \leq \frac{\pi}{b}$$
$$U = U_0 \qquad \begin{array}{l} y \leq 0 \\ y \geq \frac{\pi}{b} \end{array} \tag{2}$$

A series solution for v in the sinusoidal region is obtained in the conventional manner as shown in Appendix III. It involves two arbitrary constants which must be eliminated by the imposition of two boundary conditions.

Kuo [4] assumes that for symmetric velocity profiles and symmetric boundary conditions, there will be either symmetry or antisymmetry in the amplitude of the perturbation stream function, and therefore only half the zone need be considered. He then points out that it would be unlikely to have v = 0 along the line of maximum wind, thus reducing the study to one of symmetric disturbances.

Kuo's development is in terms of ψ , the perturbation stream function instead of v. It can be shown that, under the assumptions involved, equation (1) in either variable is identical.

The above statements result in the conclusion that v has a maximum where U is a maximum; that is,

$$v'\left(\frac{\pi}{2b}\right) = 0$$

(In this and subsequent work, the quantity in parentheses refers to the point at which the function is to be evaluated)

This equality is used as the first boundary condition and results in the elimination of one of the arbitrary constants.

Where $y < 0$, in the field of constant zonal wind, it can be shown that the solution of (1) is

$$v = K e^{ry}$$

Then

$$v' - rv = 0$$

This condition must be true at all points in the zone, therefore it is true at $y = 0$. Since $y = 0$ is common to both this zone and the sinusoidal zone, the statement is true at this point when applied to the sinusoidal zone. The second boundary condition therefore is

$$v'(0) - r\,v(0) = 0$$

The imposition of the boundary conditions is shown in Appendix III and results in a fourth degree equation in d, where $d = c - U_0$. The four roots of this equation may be evaluated, by rather tedious methods, after selection of the physical parameters involved.

Two synoptic situations in which the velocity profiles at the 500 mb. level were in fairly good agreement with that used in this development are considered as examples. These are illustrated on the following page. The parameters of the upper profile were evaluated and the equation determined for a wave length of 5000 km. This resulted in

$$d^4 - 45d^3 - 1810d^2 - 6290d - 5700 = 0$$

The roots of this equation are (approximately)

$d \approx -2$ m/sec., -1.8 m/sec., -22 m/sec., 71.5 m/sec.

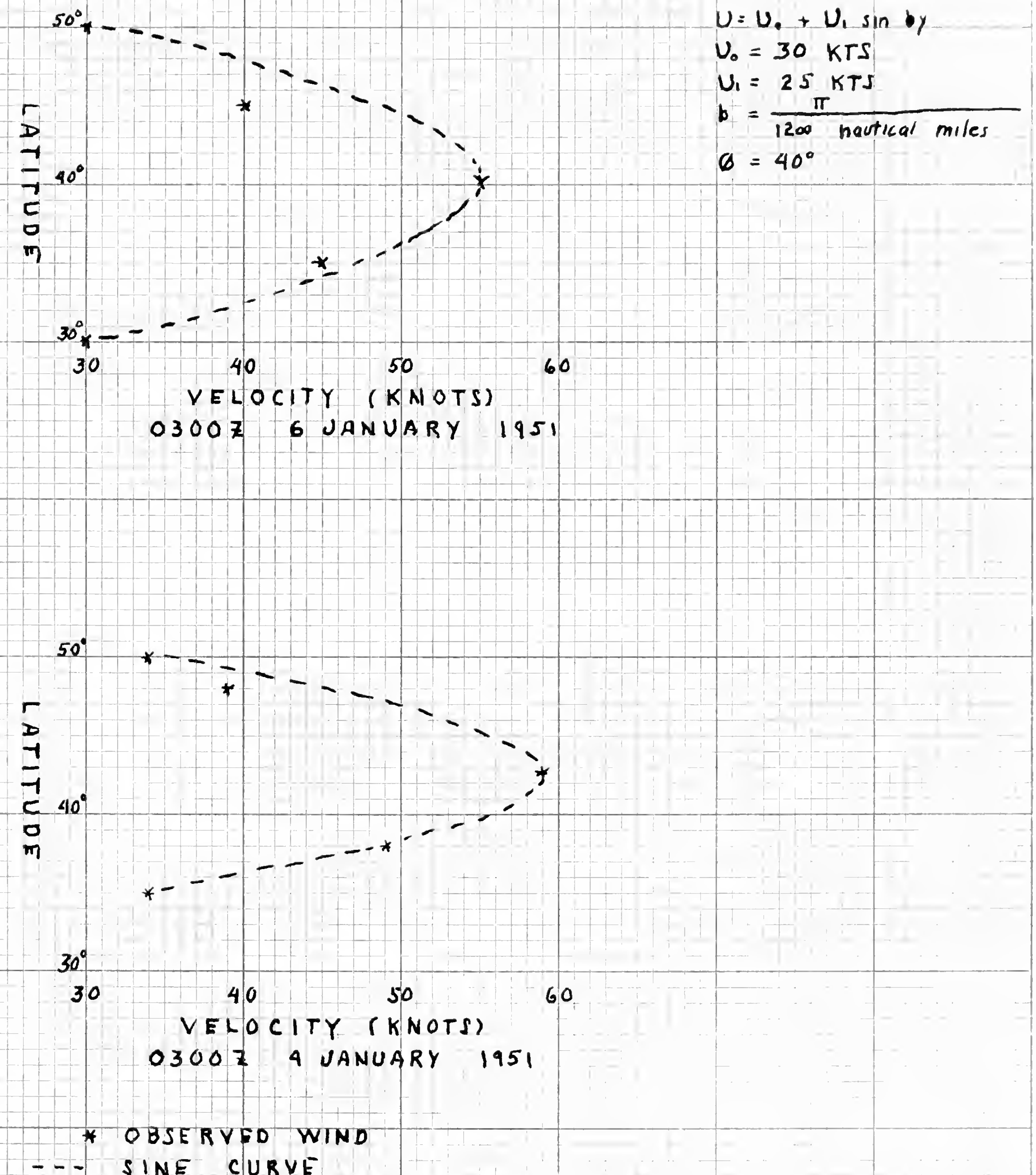

Figure 1.
500 mb. Level Velocity Profiles
Along 120° W Meridian

The last root may be discarded since we are not considering waves which move faster than minimum wind velocity. This gives the possible values of wave speed as 13.4 m/sec., 13.6 m/sec., and -6.6 m/sec. The wave is stable since all the roots are real.

III. PROBABILITY CURVE VELOCITY PROFILE

The second velocity field to be considered in the solution of (1) is

$$U = U_0 + U_1 e^{-ky^2} \tag{3}$$

By varying the parameters, this profile may be adjusted to give a good fit to a large number of symmetric profiles in the vicinity of the jet. The differential equation is solved by the series method in Appendix IV. The first boundary condition is the same as in the previous problem and when applied results in a solution for v involving one arbitrary constant.

The second boundary condition, as before, is

$$v' - rv = 0.$$

However, the "point" at which this must be applied is at infinity. The use of this condition therefore requires a transformation of the equation in which $y = \frac{1}{s}$. The point at infinity is then transformed to the origin and the boundary condition may be applied. However, the mathematics here is rather complicated and the actual expression for c in terms of α was not obtained.

IV. LINEAR SHEAR ZONE VELOCITY PROFILE

The final problem is concerned with the solution of (1) for a wind profile which is defined by the equations

$$U = U_1 + \frac{U_{11} - U_1}{l} y \qquad 0 \leq y \leq l$$

$$U = U_1 \qquad y \leq 0 \tag{4}$$

$$U = U_{11} \qquad y \geq l$$

This model differs from the previous two and from those considered by Kuo in that it is not symmetrical. It is also obviously less realistic. The solution for v in the shear zone is given in Appendix V.

By similar reasoning as in the preceding problems, the boundary conditions are found to be

$$v'(0) - r_1 v(0) = 0$$

$$v'(l) + r_{11} v(l) = 0$$

where r_1 refers to the zone of velocity U_1; and r_{11} to the zone of U_{11}.

By applying these boundary conditions as indicated in Appendix V, a rather complicated equation containing an implicit relationship between c and α results. The difficulties of finding the roots of this type equation are so great that no attempt was made to obtain numerical results for a particular model.

V. CONCLUSIONS

As with any second order differential equation with variable coefficients, the equation solved in this paper does not, in general, yield a solution in terms of elementary functions. It is quite possible that the problems considered here do possess such solutions but none was found. An attempt was made in each case to transform the equation so as to obtain a solution in terms of Bessel or Legendre functions, with no success. An arbitrary selection of wind profile will, in general, be met by the same difficulties.

The perturbation equations are, therefore, readily integrable only if the velocity profile is chosen specifically to achieve this purpose. A particular but arbitrary profile such as one of those chosen for this paper will, in general, yield an equation in which the algebraic difficulties in obtaining numerical results are of such magnitude as to make them practically worthless. These difficulties would be increased in the case of complex roots.

The restrictive assumptions made in all published studies in this field have been justified, to a certain extent, but must not be forgotten when attempting to apply results to the atmosphere. The processess of the real atmosphere are much more complex than we have assumed. The three dimensional nature of the atmosphere, divergence, vertical motion, and non-homogeneity certainly play an important role, and often a dominant one in the development and propagation of disturbances in the zonal flow.

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APPENDIX I.

DERIVATION OF SECOND ORDER ORDINARY DIFFERENTIAL EQUATION IN v AND y FROM PERTURBATION EQUATIONS

The perturbation equations for two-dimensional nondivergent flow in an incompressible fluid where the undisturbed flow is zonal are:

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + v\frac{\partial U}{\partial y} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad (5)$$

$$\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad (6)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (7)$$

In order to study the velocity of wave perturbations in the x direction, we may assume the following form of solution:

$$u = u(y)\, e^{i\alpha(x-ct)} \qquad (8)$$

$$v = v(y)\, e^{i\alpha(x-ct)} \qquad (9)$$

$$p = p(y)\, e^{i\alpha(x-ct)} \qquad (10)$$

Then from (7) and (8),

$$u = \frac{i}{\alpha}\frac{\partial v}{\partial y} \qquad (11)$$

Substituting (11) and (10) in (5) we obtain

$$-\frac{p}{\rho} = -\frac{i}{\alpha}\left[(c-U)\, v'(y)\, e^{i\alpha(x-ct)} + (U'-f)\, v\right] \qquad (12)$$

Taking the partial derivative of (12) with respect to y and equating it to the left hand member of (6), using (9) and (11), we get an equality in which each term contains the coefficient $e^{i\alpha(x-c\tau)}$. Dividing through by this quantity reduces the equation to one in which the only dependent variable is the amplitude factor of v, v(y). For sake of brevity, v(y) will now be designated as simply v. The differential equation then reduces to

$$v''(c-U) + v\left[U'' - B - \alpha^2(c-U)\right] = 0 \tag{1}$$

This may be considered as an ordinary differential equation, but it must be remembered that the final solution of v contains the periodicity factor $e^{i\alpha(x-c\tau)}$.

APPENDIX II.

VARIATION OF CORIOLIS PARAMETER WITH LATITUDE

Let the coriolis parameter be represented by

$$f = f_o + \beta y$$

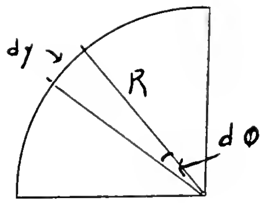
where β is assumed constant in the zone under consideration. Then $\beta = \frac{\partial f}{\partial y}$

but $f = 2 \omega \sin \Phi$

and $\frac{\partial f}{\partial y} = 2 \omega \cos \Phi \frac{\partial \Phi}{\partial y}$ (A)

From the sketch we see that

$$dy = R \, d\Phi$$



or

$$\frac{\partial \Phi}{\partial y} = \frac{1}{R}$$

Substituting this value of $\frac{\partial \Phi}{\partial y}$ in (A) we have

$$\frac{\partial f}{\partial y} = \beta = \frac{1}{R} 2 \omega \cos \Phi$$

APPENDIX III.

DEVELOPMENT OF FREQUENCY EQUATION FOR SINUSOIDAL VELOCITY PROFILE

We shall now solve the equation obtained in Appendix I for the case of a sinusoidal velocity profile:

$$U = U_0 + U_1 \sin by \qquad 0 \leq y \leq \frac{\pi}{b} \tag{13}$$

The sinusoidal profile is bounded on both sides by a field of constant velocity, U_0.

Substituting (13) in (1), we have the following equation to be solved:

$$v''(d - U_1 \sin by) + v(\Phi g \sin by + h) = 0 \tag{14}$$

where

$$d = c - U_0$$
$$\Phi = \alpha^2 - b^2$$
$$h = -B - \alpha^2 d$$

Expressing sin by in series form

$$\sin by = by - \frac{(by)^3}{3!} + \frac{(by)^5}{5!} - \cdots \tag{15}$$

and assuming a power series solution for v,

$$v = a_0 + a_1 y + a_2 y^2 + \cdots \tag{16}$$

we may substitute (15) and (16) in (14) and obtain an equation in positive powers of y equal to zero. Upon equating the coefficients of each power of y to zero, we find that

$$a_2 = -a_0 \frac{h}{2d}$$

$$a_3 = a_0 \frac{(-hbU_1 - \Phi b d U_1)}{6d^2} - a_1 \frac{hd}{6d^2}$$

Then $$v = a_0 + a_1 y - a_0 \frac{h}{2d} y^2 - \left[\frac{a_0(hbU_1 + \Phi b d U_1) + a_1 hd}{6d^2}\right] y^3 + \cdots \tag{17}$$

This expression for v involves two arbitrary constants which may be eliminated by the imposition of two boundary conditions. The first condition to be applied, as shown on page 6, is that v have a maximum where U is a a maximum, or,

$$v'\left(\frac{\pi}{2l}\right) = 0$$

Taking the first derivative of (17) with respect to y, setting $y = \frac{\pi}{2l}$ and equating the result to zero, we have

$$a_1 - a_0 \frac{h\pi}{2ld} - \left[\frac{a_0 U_1 l (h + \omega d) + a_1 hd}{2d^2}\right]\frac{\pi^2}{4l^2} = 0 \qquad (18)$$

and upon solving for a_1 we see that

$$a_1 = a_0 \frac{4\pi h l d + U_1 l \pi^2 (h + \omega d)}{8l^2 d^2 - hd\pi^2} \qquad (19)$$

The resulting solution for v, neglecting, for the moment, the third and higher powers of y, is

$$v = a_0 \left[1 + \frac{4\pi h l d + U_1 l (h + \omega d)\pi^2}{8l^2 d^2 - hd\pi^2} y - \frac{h}{d} y^2\right] \qquad (20)$$

By imposing the second boundary condition (see page 7),

$$v'(0) - r\, v(0) = 0$$

we obtain

$$a_0 \frac{4\pi h l d + U_1 l (h + \omega d)\pi^2}{8l^2 d^2 - hd\pi^2} - a_0 \left(\alpha^2 + \frac{B}{d}\right)^{1/2} = 0 \qquad (21)$$

This step shows that only the first two terms of the series solution for v are significant in the development of the frequency equation so the powers

of y higher than the first could have been ignored after having determined equation (19). However, it should be noted that the solution of a_1 in equation (19) is approximate since only the first four terms of the general solution for v were used in its determination. It follows that the frequency equation is approximate.

Squaring equation (21) and collecting the coefficients of powers of d we have

$$
\begin{aligned}
& d^4\left(-64 l^4 d^2 - \pi^4 \alpha^6\right) - d^3\left(8\pi^3 l^4 \alpha^2 g - 3\pi^4 \alpha^4 \beta\right) \\
& + d^2\left(8\pi^3 l^4 g \beta + \pi^4 g^2 l^6 + 8\pi^3 l^2 \alpha^2 \beta - 2\pi^4 \alpha^2 \beta^2 - \pi^2 l^2 \alpha^2\right) \\
& + d\left(8\pi^3 l^2 \beta^2 g + 2\pi^4 l^4 g^2 \beta - \pi^4 \beta^3\right) + \pi^4 g^2 l^2 \beta^2 = 0
\end{aligned}
\qquad (22)
$$

APPENDIX IV.

DEVELOPMENT OF FREQUENCY EQUATION FOR PROBABILITY CURVE VELOCITY PROFILE

In this section, the differential equation

$$v''(c-U) + v(U'' - \beta - \alpha^2 c + \alpha^2 U) = 0 \tag{1}$$

is to be solved for the case in which the undisturbed wind velocity profile is defined by the probability curve

$$U = U_0 + U_1 e^{-ky^2} \tag{23}$$

A power series solution for v is assumed,

$$v = a_0 + a_1 y + a_2 y^2 + \cdots\cdots \tag{24}$$

and e^{-ky^2} is expressed in series form,

$$e^{-ky^2} = 1 - ky^2 + \frac{k^2 y^4}{2!} - \frac{k^3 y^6}{3!} + \cdots\cdots \tag{25}$$

The series expressions for v and U are then substituted in (1). The coefficients of each power of y are equated to zero and we may now obtain expressions for any a_n in terms of a_o and a_1:

$$a_2 = a_0 \frac{\alpha^2 U_m - 2kU_1 - \alpha^2 c - \beta}{2(U_m - c)}$$

$$a_3 = a_1 \frac{\alpha^2 U_m - 2kU_1 - \alpha^2 c - \beta}{6(U_m - c)}$$ where $U_m = U_0 + U_1$ and

a_o and a_1 are arbitrary constants. The general solution for v is therefore

$$v = a_0 + a_1 y + a_0 \frac{\alpha^2 U_m - 2kU_1 - \alpha^2 c - \beta}{2(U_m - c)} y^2 + a_1 \frac{\alpha^2 U_m - 2kU_1 - \alpha^2 c - \beta}{6(U_m - c)} y^3 + \cdots\cdots \tag{26}$$

We may eliminate one of the constants by the imposition of the boundary condition

$$v'(0) = 0$$

The use of this condition for symmetrical velocity profiles is discussed on page 6.

Taking the first derivative of (26) with respect to y, and setting y = 0, we obtain

$$a_1 = 0$$

Equation (26) then reduces to

$$v = a_0 \left[1 + \frac{\alpha^2 U_m - 2k U_1 - c\alpha^2 - \beta}{2(U_m - c)} y^2 + \cdots \right.$$

As mentioned on page 10, the application of the second boundary condition,

$$v'(-\infty) - r\, v(-\infty) = 0$$

leads to mathematical difficulties and the elimination of a_0 and consequent development of the frequency equation was not accomplished.

APPENDIX V.

DEVELOPMENT OF FREQUENCY EQUATION FOR A LINEAR SHEAR ZONE

In this section the differential equation

$$v''(c-U) + v(U'' - \beta - \alpha^2 c + \alpha^2 U) = 0 \qquad (1)$$

is solved for the case in which the undisturbed wind velocity profile is a linear shear zone. The arbitrary constants are eliminated and the frequency equation developed for the case in which the shear zone is terminated, on each side by a field of constant zonal wind.

The shear zone is defined by

$$U = U_1 + ky \qquad \text{where } k = \frac{U_{11} - U_1}{l} \qquad (27)$$

A power series solution for v is assumed

$$v = a_0 + a_1 y + a_2 y^2 + \cdots$$

and this series is substituted in (1). The coefficients of each power of y are equated to zero and the following relationships obtained;

$$a_2 = a_0 \frac{\alpha^2 d + \beta}{2d}$$

$$a_3 = a_0 \frac{k\beta}{6d^2} + a_1 \frac{d(\alpha^2 d + \beta)}{6d^2} \qquad \text{where } d = c - U_1.$$

The series solution for v is then

$$v = a_0 + a_1 y + a_0 \frac{\alpha^2 d + \beta}{2d} y^2 + \left[a_0 \frac{k\beta}{6d^2} + a_1 \frac{d(\alpha^2 d + \beta)}{6d^2}\right] y^3 + \cdots \qquad (28)$$

As explained on page 11, the boundary conditions to be imposed are

$$v'(0) - \lambda_1 v(0) = 0 \qquad (29)$$

$$v'(l) + \lambda_{11} v(l) = 0 \qquad (30)$$

By substituting (28) in (29) and letting y = 0 we find that

$$a_1 = r_1 a_0$$

Equation (28) then reduces to

$$v = a_0 \left[1 + r_1 y + \frac{\alpha^2 d + \beta}{2d} y^2 + \frac{\kappa\beta + r_1 d(\alpha^2 d + \beta)}{6d^2} y^3 + \cdots \right] \tag{31}$$

To impose the second boundary condition, we substitute (31) in (30) and set $y = \ell$. After dividing through by the common factor a_0, we obtain the frequency equation which becomes, upon rearranging:

$$\begin{aligned} &6d^2(r_1 + r_{11} + r_1 r_{11} \ell) + 6d\ell(\alpha^2 d + \beta) \\ &+ 3d r_{11} \ell^2(\alpha^2 d + \beta) + 3\ell^2\left[\kappa\beta + r_1 d(\alpha^2 d + \beta)\right] \\ &+ r_{11} \ell^3\left[\kappa\beta + r_1 d(\alpha^2 d + \beta)\right] = 0 . \end{aligned} \tag{32}$$

This is approximate since fourth and higher powers of y have been neglected.

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